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Iterative Computation of Modal Sensitivities

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The computation of frequency and mode shape sensitivities with respect to design parameters is essential to many structural optimization and finite element updating algorithms. Approximating the gradients may lead to poor estimates and loss of convergence, but on the other hand the cost of exact methods, such as Nelson's (Nelson, R., "Simplified Calculations of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1201–1205), is often not acceptable for industrial-size models. The present study thus analyses existing approximation methods, suggests a first-order improvement of Fox and Kapoor's modal basis (Fox, R., and Kapoor, M., "Rate of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429) and proposes a residual iteration technique allowing accurate computations of modeshape sensitivities. An engine cover model and a model of the Ariane 5 main cryogenic stage are used to analyze, for models of realistic sizes, the accuracy and computational costs of the proposed methods.

Nomenclature

K	=	stiffness matrix
$[\hat{K}]^{-1}$	=	pseudo-inverse of the stiffness matrix
M	=	mass matrix
p	=	vector of model parameters
$R_{j,D}$	=	displacement residual for a sensitivity estimate
$R_{j,L}$	=	load residual for a sensitivity estimate
T	=	reduction basis
Z	=	dynamic stiffness matrix
δ_{jk}	=	Kronecker symbol
ϵ_j	=	relative energy error
ϕ_j	=	mode shape
ψ_j	=	particular solution of sensitivity equation
ω_j	=	modal frequency

Subscript

j	=	mode number

Superscript

(n)	=	basis number in iterative refinement

I. Introduction

OPTIMIZATION methods are widely used for finite element model updating and structural optimization. Most of these approaches use partial derivatives, called *sensitivities*, of properties with respect to physical parameters of the model either to select sensitive parameters or to orient the search for a minimum of various objective functions. Accurate and yet inexpensive evaluations of sensitivities are thus a major issue.

Computation of eigenvalue and eigenvector sensitivities has been the object of an extensive literature. The modal method[1] is the most widespread although its relatively poor accuracy is well known. A method to compute the exact solution was proposed by Nelson,[2] but the associated computational cost is often excessive for industrial models. As alternates to the exact method, Ojalvo and Zhang[3] proposed to use a basis of Lanczos vectors to replace the modes used by Fox and Kapoor.[1] Wang[4] and in a more general setting Liu and Zhao[5] proposed the use of static corrections to the modal method, and Alvin[6] introduced a conjugate gradient-type method to compute the complementary part of the modal contribution.

Section II details Nelson's method and gives a presentation of the aforementioned approximate methods within the unified framework of model reduction proposed by Balmès.[7] This framework allows the simple introduction of new approaches, especially the residual iteration method, to obtain low-cost predictions of both mode shapes and their sensitivities. The residual iteration method combines classical ingredients[8]: reduced solution in a subspace, error evaluation using an energy-based criterion on displacement residuals (which can also be seen as error evaluation on preconditioned residuals), and basis enrichment with an orthonormalization procedure. The paper focuses on modal sensitivities, but this methodology can be applied with almost no modification to a range of other problems: computing modes of elastic or damped structures, coupled or not with a fluid,[9] computing minimum dynamic residual expansion,[10] frequency response function sensitivities, finite/boundary element coupling problems,[11] and so on.

Section III presents a detailed analysis of the various methods in terms of accuracy and computational cost for a 16,840-degree-of-freedom (DOF) model of an engine cover and a 36,928-DOF model of the Ariane 5 main cryogenic stage. These models provide real insight on the applicability of the methods to industrial problems while still allowing the computation of an exact solution for accuracy evaluations.

II. Theoretical Aspects

Methods to compute modal sensitivities can be either exact or approximate. This section presents the Nelson's exact method[2] for cases of single and multiple eigenvalues, and introduces a unified framework for approximate methods. Approximate methods are then divided into direct and iterative approximations, and the proposed residual iteration technique is introduced.

A. Exact Solution for Single Eigenvalue

Modes are solutions of the eigenvalue problem

$$\left[K(p) - \omega_j^2 M(p)\right]\{\phi_j\} = [Z(\omega_j, p)]\{\phi_j\} = \{0\} \tag{1}$$

and satisfy two orthogonality conditions with respect to mass,

$$\{\phi_k\}^T [M(p)]\{\phi_j\} = \delta_{jk} \tag{2}$$

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and stiffness,

$$\{\phi_k\}^T[K(p)]\{\phi_j\} = \omega_j^2\delta_{jk} \tag{3}$$

The mass normalization of mode j, linked to the constant $\{\phi_j\}^T[M]\{\phi_j\}$, is arbitrary and will be assumed to be equal to 1 for all values of p [as shown in the orthogonality conditions (2) and (3)].

Equation (1) being valid for all values of design parameter p, its derivative with respect to p is also equal to zero, which results in

$$[Z(\omega_j, p)]\left\{\frac{\partial\phi_j}{\partial p}\right\} = \{B(\omega_j, p)\} \tag{4}$$

where

$$B(\omega_j, p) = \left[-\frac{\partial K}{\partial p} + \frac{\partial\omega_j^2}{\partial p}M(p) + \omega_j^2\frac{\partial M}{\partial p}\right]\{\phi_j\} \tag{5}$$

By definition of modes (1), the dynamic stiffness $[Z(\omega)]$ is singular at modal frequencies ω_j so that Eq. (4) does not necessarily have a solution. A well-known theorem of linear algebra stipulates that equations of the form $Zq = B$ with Z singular have solutions if, and only if, B is orthogonal to the kernel of Z^T. From Eq. (1), this kernel is the mode $\{\phi_j\}$ in the case of a single eigenvalue and the modal subspace for multiple eigenvalues (as treated in Sec. II.B). One must thus have $\{\phi_j\}^T B(\omega_j) = 0$, which defines the modal frequency sensitivities

$$\frac{\partial\omega_j^2}{\partial p} = \{\phi_j\}^T\left[\frac{\partial K}{\partial p} - \omega_j^2\frac{\partial M}{\partial p}\right]\{\phi_j\} \tag{6}$$

Again, it is known from linear algebra that solutions of Eq. (4) take the general form $\{\partial\phi_j/\partial p\} = \psi_j + \alpha\phi_j$, where ψ_j is an arbitrary particular solution of $[Z(\omega_j)]\{\psi_j\} = \{B(\omega_j)\}$.

As proposed by Nelson,[2] a particular solution of Eq. (4) can be determined by imposing that one of the components of ψ_j is equal to zero. This particular solution clearly exists as long as the corresponding component of ϕ_j is nonzero. Knowing that a component of ψ_j is zero, one can eliminate a row and a column of Eq. (4) that leads to a nonsingular set of equations that can be solved relatively easily. This solution however requires the factorization of a block of $Z(\omega_j)$. This factorization must be performed at the frequency of each of the desired mode shape sensitivities, which tends to be very expensive for realistic finite element models. (See numerical applications in Sec. III.)

Finally a condition is needed to define the coefficient α in the general form of the solution. Assuming that the mode shape is always mass normalized as shown in Eq. (2), this condition can be derived with respect to p, which leads to

$$\{\phi_j\}^T M\left\{\frac{\partial\phi_j}{\partial p}\right\} = -\frac{1}{2}\{\phi_j\}^T\frac{\partial M}{\partial p}\{\phi_j\} \tag{7}$$

Thus, given ψ_j a particular solution of $Z(\omega_j)\psi_j = B(\omega_j)$, the sensitivity of the mass normalized mode shapes is given by

$$\left\{\frac{\partial\phi_j}{\partial p}\right\} = \{\psi_j\} - \left(\phi_j^T M\psi_j + \frac{1}{2}\phi_j^T\frac{\partial M}{\partial p}\phi_j\right)\{\phi_j\} \tag{8}$$

B. Multiple Eigenvalue

In the case when ω_j has a multiplicity $k > 1$, the associated eigenvectors, solutions of Eq. (1), lie in a subspace of dimension k. Let $[\Phi]_{N\times k}$ (N the number of DOFs) be a basis of this subspace, built orthonormal with respect to the mass, i.e., $[\Phi]^T[M][\Phi] = I_{k\times k}$. Every vector $\{\phi_i\}$ of this subspace is an eigenmode for the eigenfrequency ω_j and can be written as

$$\{\phi_i\} = \sum_{l=1}^{k} c_{li}\{\Phi_l\} = [\Phi_1\cdots\Phi_k]\{c_i\} \tag{9}$$

or in matrix form

$$[\phi]_{N\times k} = [\Phi]_{N\times k}[c]_{k\times k} \tag{10}$$

where $[\phi]$ is a matrix containing the $\{\phi_i\}$ and $[c]$ is orthogonal ($c^T c = I$).

Noting $[B(\omega_j)]_{N\times k}$ the matrix containing the vectors

$$[B(\omega_j)]_i = \left[-\frac{\partial K}{\partial p} + \frac{\partial\omega_j^2}{\partial p_i}M + \omega_j^2\frac{\partial M}{\partial p}\right]\{[\Phi][c]\}_i \tag{11}$$

the equation giving sensitivities has a solution if and only if $[B(\omega_j)]_{N\times k}$ is orthogonal to the kernel of $Z(\omega_j)^T$. Thus here one must have $[\Phi]^T[B(\omega_j)] = 0$, which leads to a $k\times k$ eigenvalue problem

$$\left([\Phi]^T\left[\frac{\partial K}{\partial p} - \omega_j^2\frac{\partial M}{\partial p}\right][\Phi]\right)[c]_i = \frac{\partial\omega_j^2}{\partial p_i}[c]_i \tag{12}$$

The resulting eigenvalues are the eigenfrequency sensitivities $\partial\omega_j^2/\partial p_i$, and the resulting eigenvectors are vectors $[c]_i$ giving the particular eigenmodes $[\phi]_i$ for which the sensitivities are computed.

Mode shape sensitivities are then of the form

$$\left[\frac{\partial\phi}{\partial p}\right]_{N\times k} = [\psi] + [\phi][\alpha]_{k\times k} \tag{13}$$

where $[\psi]_{N\times k}$ contains the arbitrary particular solutions of $[Z(\omega_j)]\{\psi_i\} = [B(\omega_j)]_i$. These particular solutions are computed by deleting k columns and vectors of $Z(\omega_j)$ on DOFs that do not correspond to null values of the corresponding $[\phi]$. In practice, one deletes the rows and columns corresponding to the maximum amplitudes of each $[\phi]_i$. (A row can be deleted only once.)

If for a particular i, $\partial\omega_j^2/\partial p_i$ has a multiplicity $k' > 1$, then the $[c]_i$ are still undetermined. Shaw and Jayasuriya[12] treated this case by determining $[c]_i$ and the corresponding α, considering higher order derivatives of ω_j^2.

Note that all derivatives may be multiple (for example, if the design parameter of interest respects the cyclic symmetry of a structure) and $[c]_i$ is then arbitrary. In this case, Dailey[13] proposed a procedure for the determination of $[c]_i$ and α.

C. Reduced Basis Approximations

Reduction methods are widely used to seek approximations of the properties of dynamic systems. They are all based on the assumption that an accurate approximation of the response can be found in the subspace spanned by the columns of a rectangular reduction basis T (with N rows and $NR \ll N$ columns). The approximate modes of a model reduced on basis T are given by $\{\phi_j\}_{\text{approx}} = T\{\phi_j\}_R$ with $\{\phi_j\}_R$ solution of the reduced eigenvalue problem

$$[T]^T\left[K(p) - \omega_{jR}^2 M(p)\right][T]\{\phi_j\}_R = \{0\} \tag{14}$$

Assuming that T is fixed, Eq. (14) is valid for all values of p and can be derived as done for the full-order model in the preceding section. The general form of the approximate sensitivity is then given by

$$\left\{\frac{\partial\phi_j}{\partial p}\right\}_{\text{approx}} = [T]\left\{\frac{\partial\phi_j}{\partial p}\right\}_R = [T]\{\psi_j\}_R + \alpha T\{\phi_j\}_R \tag{15}$$

where $\{\psi_j\}_R$ is solution of

$$[Z_R(\omega_j)]\{\psi_j\}_R = \{B(\omega_j)\}_R \tag{16}$$

with $Z_R = T^T Z T$ and

$$\{B(\omega_j)\}_R = T^T\left[-\frac{\partial K}{\partial p} + \frac{\partial\omega_j^2}{\partial p}M + \omega_j^2\frac{\partial M}{\partial p}\right]\{\phi_j\} \tag{17}$$

When designing improved methods for the approximation of sensitivities, the effort should thus concentrate on building a reduction basis that will give accurate predictions of the sensitivities. This can be done using a direct approximation (see Sec. II.D), that is, by building a reduction basis that is thought to be accurate and using it for the resolution of Eq. (16) or using an iterative approximation (see Sec. II.E), that is, by adapting the reduction basis to minimize an error indicator.

D. Direct Approximations

Direct approximations have been intensively studied. The most widespread reduction basis is the truncated modal basis,[1] which although generally available, is not the most efficient in terms of allowing accurate predictions of modeshape sensitivities.

Ojalvo and Zhang[3] realized that the estimates of $\phi_j(p)$ are often determined by projection of the model on a basis of Lanczos vectors that span the same subspace as the Krylov vectors given by $T^{(n)} = (K^{-1}M)^p T^{(0)}$ and thus proposed a method allowing the use of the same basis of Lanczos vectors for the estimation of mode shapes and mode shape sensitivities.

This approach continues to accept the cost of computing the exact modes at each iteration but uses a larger projection basis to estimate sensitivities, so that results can be expected to be more accurate.

Realizing that in the sensitivity equation (4) the right-hand-side term $B(\omega_j)$ corresponds to a load that is representative of the modification, Balmès[14] completed the modal basis of Fox's method by the static responses to these loads:

$$T_C = \left[[\phi_1(p)\cdots\phi_{NM}(p)][\hat{K}]^{-1}[B(\omega_i, p)]_{i\in T}\right] \tag{18}$$

where $1\ldots NM$ are the low-frequency modes containing all target modes (in set T) for which the sensitivities are desired. The usefulness to keep more modes than desired will be illustrated in Sec. III.C.

As mentioned by Liu and Zhao,[5] a mass shifted stiffness matrix $\hat{K} = K + \lambda M$ can be used in place of the nominal stiffness matrix when rigid body modes pose problems. The static flexible response would be another alternative. (See Sec. 6.8 of Ref. 15 on iterations in the presence of rigid body modes.)

This first-level correction can be extended using the vectors of the Krylov series $T^{(n)} = [\hat{K}^{-1}M]^k[\hat{K}]^{-1}[B(\omega_i, p)]_{i\in T}$, but care must be taken to orthogonalize the successive additions $T^{(n)}$ to the subspace spanned by T_C. One could, for example, use the Lanczos orthogonalization scheme for this purpose. This would go back to the idea proposed by Ojalvo and Zhang[3] but uses $[\hat{K}]^{-1}[B(\omega_i, p)]_{i\in T}$ as the base vectors to start a block Lanczos algorithm.

Note that one saves a lot of time by using the same factorization of K or $\hat{K}$ to compute eigenvectors (using a subspace or Lanczos method) and the corrections (18) needed to compute accurate sensitivities.

E. Iterative Approximations

A number of papers, in particular Refs. 16 and 17, discuss iterative techniques related to subspace iteration eigenvalue solvers. Their idea is to use Fox and Kapoor's[1] approximation as a starting point and to estimate the contributions of unavailable modes using a converging iteration operator.

As illustrated in Fig. 1, subspace iteration methods have the advantage of using a fixed number of vectors. However, Lanczos methods, which gradually build a basis without size limitations, typically converge faster for iterative eigenvalue computations. The residual iteration procedure proposed here uses a Lanczos-like strategy of basis enrichment, where residuals are computed to evaluate the accuracy of the reduced solution (15) and, if needed, displacement residuals are added to the reduction basis. The considered load residuals are defined by

$$\{R_{j,L}\} = [Z(\omega_j)]T\left\{\frac{\partial\phi_j}{\partial p}\right\}_R - B(\omega_j) \tag{19}$$

which is not zero because $T\{\partial\phi_j/\partial p\}_R$ does not satisfy Eq. (4). To obtain an energy-based error indicator, one computes displacement residuals with

$$\{R_{j,D}\} = [\hat{K}]^{-1}\{R_{j,L}\} \tag{20}$$

Starting from a basis $T^{(0)}$ obtained with one of the methods in Sec. II.D, the following steps are performed at iteration n: 1) computation of the load residuals $\{R_{j,L}^{(n)}\}$ associated with basis $T^{(n)}$ using Eq. (19), 2) computation of the displacement residuals using Eq. (20), 3) evaluation of the relative strain energy error indicator (of course a kinetic energy criterion could alternatively be defined, or a mix of both)

$$\epsilon_j^{(n)} = \frac{\left\|\left\{R_{j,D}^{(n)}\right\}\right\|_K^2}{\left\|T^{(n)}\{\partial\phi_j/\partial p\}_R^{(n)}\right\|_K^2} \tag{21}$$

and 4) basis completion: $\epsilon_j^{(n)} > \text{Tol} \Rightarrow T^{(n+1)} = [T^{(n)}, \{R_{j,D}^{(n)}\}]$. These steps are repeated until every ϵ_j is less than a user-fixed tolerance (typically 10^{-6}).

It should be noted that the projection at step n of $[Z(\omega_j)]$ needed for the resolution of Eq. (16) can be accelerated using the projection at step $n-1$. Furthermore, normalization problems arise from adding residuals to the basis $T^{(n)}$, which requires the use of an orthonormalization procedure as in Lanczos algorithms.

A basic subspace iteration method (without the acceleration technique proposed by Ting[17] will be used for comparisons in Sec. III. The difference between the methods is that for subspace iterations, the basis retained in step 4 is given by

$$T^{(n+1)} = \left[\phi_{1\ldots NM}, T^{(n)}\left\{\frac{\partial\phi_j}{\partial p}\right\}_R^{(n)}, R_D^{(n)}\right] \tag{22}$$

where one retains the exact modes, the last estimate of the sensitivities, and the displacement residuals that are above the tolerance. In practice, the sensitivities and residues need to be orthogonalized with respect to the modes retained to avoid numerical conditioning problems linked to strong vector collinearity, a step that was inherent in the formulations proposed in Refs. 16 and 17.

Comparisons are also made with the preconditioned conjugate projected gradient (PCPG)-based technique proposed by Alvin. [6]

As for subspace methods, the PCPG method starts from Fox and Kapoor's[1] approximation and computes the complementary part of the modal contribution using a conjugate gradient-type method. The PCPG method allows us to compute the sensitivity of one mode with respect to one parameter and must be restarted for each mode/parameter set.

Starting from the modal approximation, i.e.,

$$\left\{\frac{\partial\phi_j}{\partial p}\right\} \approx \sum_{k=1}^{NM} a_{j,k}\{\phi_k\} = [\phi]a_j \tag{23}$$

one computes the complementary part v_j, such that

$$Z(\omega_j)v_j = B(\omega_j) - Z(\omega_j)[\phi]a_j \tag{24}$$

$Z(\omega_j)$ is not positive definite, but if one notes $[\hat{\phi}]$ the matrix containing the modes $NM+1$ to N (N being the number of DOFs), then the projected problem

$$\left([\hat{\phi}]^T Z(\omega_j)[\hat{\phi}]\right)x_j = [\hat{\phi}]^T(B(\omega_j) - Z(\omega_j)[\phi]\boldsymbol{a}_j) \tag{25}$$

with $v_j = [\hat{\phi}]x_j$, uses a positive definite symmetric matrix (for $j \le NM$) and can be solved by the conjugate gradient method.

The $[\hat{\phi}]$ being unknown a priori (only the first modes are computed), Alvin[6] uses the relation

$$M^{-1} = [\phi, \hat{\phi}][\phi, \hat{\phi}]^T \tag{26}$$

to transform Eq. (25). The obtained projected problem is solved by a conjugate gradient-type method, using the factorization of K (or $\hat{K}$) as a precondioner. As for the other methods, the same factorization can be used to compute the modes and the sensitivities.

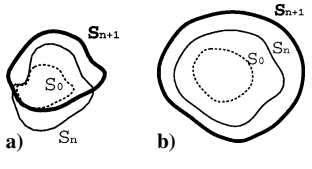

Fig. 1 Representation of a) subspace and b) residual iteration.

The convergence criterion proposed by Alvin is the classical conjugate gradient criterion based on the quadratic norm of the residual. When comparing the CPU times of various methods, the same convergence criterion must be used, so that in Sec. III criterion (21) is used for all of the considered methods.

III. Applications

This section presents a detailed analysis of the various methods for two cases detailed in Sec. III.A. Section III.B analyzes the accuracy of the different approximate methods using Nelson's exact results. In Sec. III.C, the relative effects of the number of modes in the initial basis T^0, the number of parameters considered, and the number of target modes are analyzed to show computational cost tradeoffs.

A. Cases Studied

The first application uses an engine cover model of 16,840 DOFs studied in free–free conditions. Except for importing the element matrix dictionary of the engine cover from NASTRAN, all computations were performed using the Structural Dynamics Toolbox[18] for MATLAB. Ten parameters are studied, namely, the stiffness and mass of the five groups of elements described in Fig. 2. The second application concerns a model of the Ariane 5 rocket main cryogenic stage (Fig. 3) with 29 superelements, which assemble in a model with 36,928 DOFs. Again superelement matrices were imported into the Structural Dynamics Toolbox in which all computations were performed.

The 10 design parameters are stiffness related to elements belonging to the oxygen line (in bold in Fig. 3).

B. Accuracy

First, the convergence of the iterative process to the exact sensitivities needs to be checked to validate convergence criterion (21). This is done by computing the relative energy error (strain and kinetic energy) for each sensitivity $\partial\phi_j/\partial p_i$ (j in the range of T, T being the indices of the target eigenvectors for which the sensitivities are desired, and i the index of the parameter), that is

$$\alpha_{i,j} = \frac{\left\| \partial\phi_j \big/ \partial p_{i\,\mathrm{exa}} - \partial\phi_j \big/ \partial p_{i\,\mathrm{approx}} \right\|_K^2}{\left\| \partial\phi_j \big/ \partial p_{i\,\mathrm{exa}} \right\|_K^2 + \left\| \partial\phi_j \big/ \partial p_{i\,\mathrm{approx}} \right\|_K^2} \tag{27}$$

$$\beta_{i,j} = \frac{\left\| \partial\phi_j \big/ \partial p_{i\,\mathrm{exa}} - \partial\phi_j \big/ \partial p_{i\,\mathrm{approx}} \right\|_M^2}{\left\| \partial\phi_j \big/ \partial p_{i\,\mathrm{exa}} \right\|_M^2 + \left\| \partial\phi_j \big/ \partial p_{i\,\mathrm{approx}} \right\|_M^2} \tag{28}$$

where $\partial\phi_j/\partial p_{i\,\mathrm{exa}}$ is the sensitivity computed by Nelson's method and $\partial\phi_j/\partial p_{i\,\mathrm{approx}}$ is the sensitivity computed on reduced bases. Here three bases are considered: the modal basis (Fox and Kapoor's method) containing the NM first modes, where NM is the maximum index of the target modes; the first-order static correction T_C given in Eq. (18); and the basis enriched by the residual iteration (Sec. II.E), starting from basis T_C [Eq. (18)] with $\mathrm{Tol} = 10^{-5}$. Figures 4 and 5 represent relative energy errors on sensitivities (see legends) for the engine cover and the Ariane 5 rocket, respectively. From these figures, criterion (21) used for enrichment is relevant because a tolerance of 10^{-5} gives a relative strain-kinetic energy error associated to the iterative results of the same order. Results obtained with a modal basis are only rough approximations of the

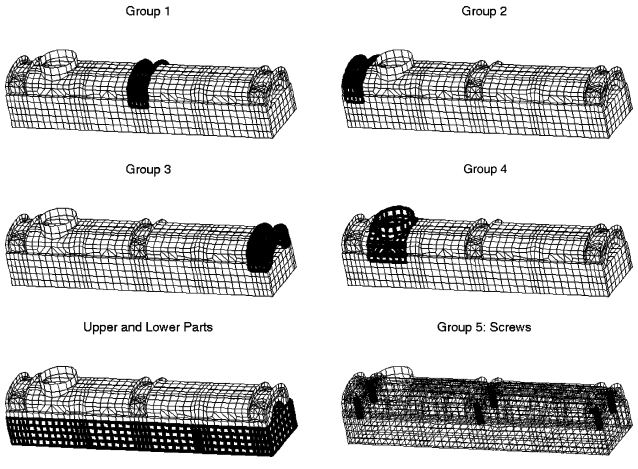

Fig. 2 Groups 1–5; group 5 consists of six screws connecting the upper and lower parts of the engine cover.

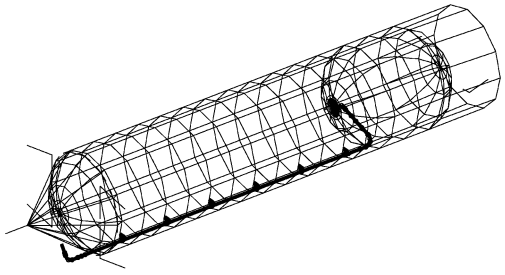

Fig. 3 Visualization model of the main cryogenic stage of the Ariane 5 rocket; the oxygen line is in bold.

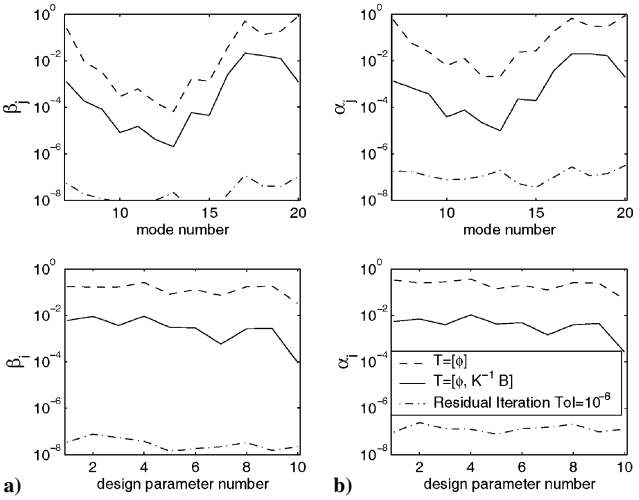

Fig. 4 Relative energy errors on sensitivities for different reduction bases (engine cover): a) kinetic and b) strain energy errors averaged over the 10 directions for modes 7–20 and over the 14 modes for parameters 1–10.

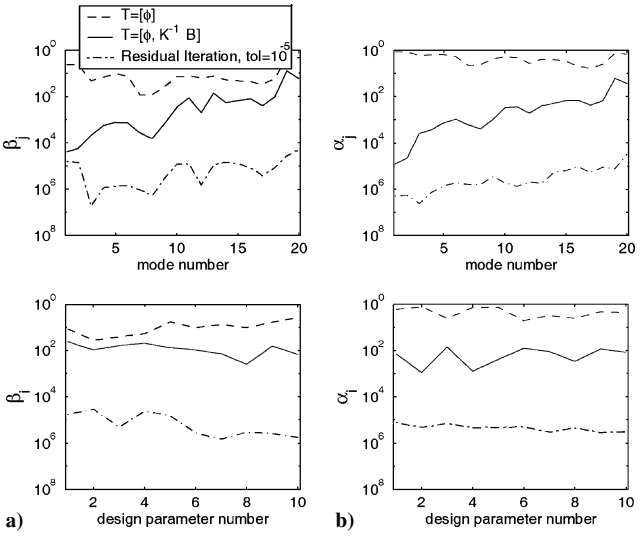

Fig. 5 Relative energy errors on sensitivities for different reduction bases (Ariane 5): a) kinetic and b) strain energy errors averaged over the 10 directions for modes 1–20 and over the 20 modes for parameters 1–10.

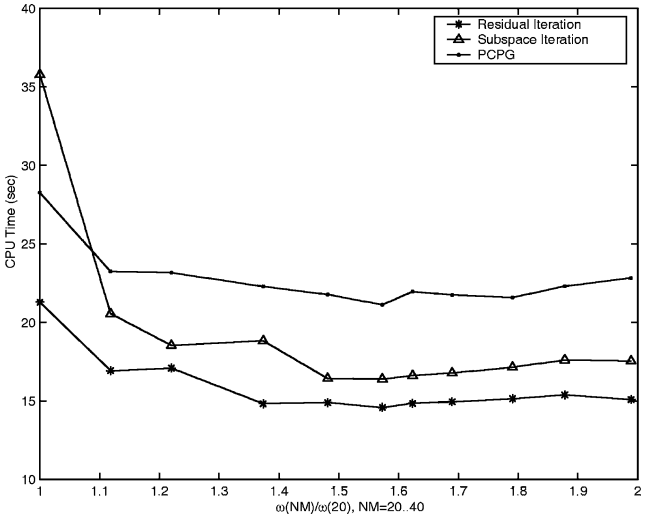

Fig. 6 Engine cover, modes 7–20, 10 parameters; computational time vs $\omega_{NM}/\omega_{\max(T)}$ [$NM = 20 \ldots 40$, $\max(T) = 20$].

exact sensitivities (the relative error reaches 100% at one point) and using basis T_C, although greatly improving the approximation, still lacks accuracy (relative error reaching 10%).

It should be noted that residual iteration converges for every tolerance ($\mathrm{Tol} = 10^{-12}$ has been tested as a minimum value for the engine cover), with no particular conditioning problems arising.

C. Computational Tradeoffs

The next results concern computational time aspects. Before going further, it should be noted that CPU time is only an indicator and should not be considered as an absolute measure. Indeed, even when running on a single machine (here a Pentium III biprocessor at 1 GHz) there are visible variations between runs.

Moreover, when comparing CPU times of various methods, the same convergence criterion must be used, so that in this paper criterion (21) is used for all methods. Because the evaluation of the criterion used by the PCPG method, based on the quadratic norm of a residual, is much less expensive than the evaluation of Eq. (21), the time spent for the evaluation of Eq. (21) will be deduced from the CPU times related to the PCPG method.

Another distortion is linked to the level of optimization. On modern machines, careful selection of numerical libraries (Basic Linear Algebra Subroutines, sparse libraries, etc.) and memory allocation can lead to major changes in computational speed. Although we

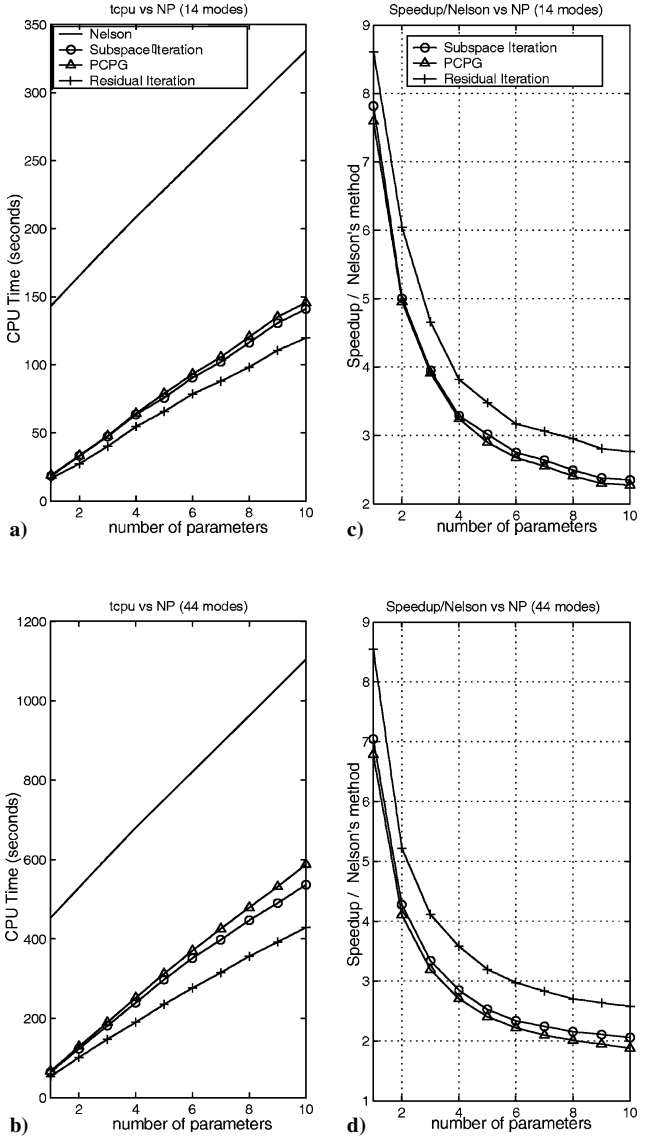

Fig. 7 Influence of design parameters (engine cover model): CPU time for modes a) 7–20 and b) 7–05 and speedup with respect to Nelson's method for modes c) 7–20 and d) 7–05.

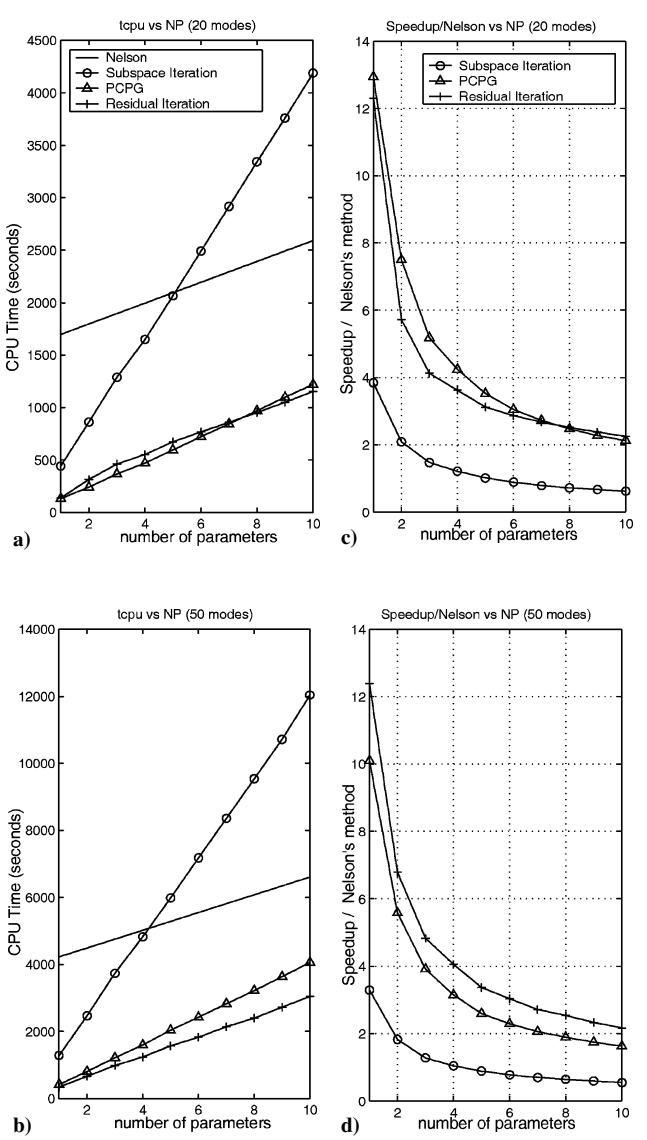

Fig. 8 Influence of design parameters (Ariane 5 model): CPU time for modes a) 7–20 and b) 7–05 and speedup with respect to Nelson's method for modes c) 7–20 and d) 7–05.

were careful, the implementation could certainly be improved significantly. The trends illustrated in this section would, however, remain true.

The first aspect of interest is the effect of the number of modes in the initial approximation (initial basis $T^0 = T_C$ for subspace and residual iteration, or Fox and Kapoor approximation for the PCPG method). Figure 6 represents the computational time required for the various methods studied to reach a tolerance of 10^{-6} vs the frequency ratio of the last mode retained in the initial approximation to the last mode for which the sensitivity is to be computed, $\omega_{NM}/\omega_{\max(T)}$.

For $\omega_{NM}/\omega_{\max(T)} < 1.5-1.6$, CPU time decreases with increasing NM, and for $\omega_{NM}/\omega_{\max(T)} > 1.6$, CPU time slightly increases with increasing NM. This shows that a compromise must be reached. Indeed, using more modes in the first approximation increases its accuracy, but using too many requires preliminary computations that are costly in terms of CPU and memory. Because the slope is higher for $\omega_{NM}/\omega_{\max(T)} < 1.2$, it seems that using in the initial approximation the modes until $\omega_{NM}/\omega_{\max(T)} \approx 1.2$ is a good compromise. This criterion will be applied to build the initial approximation in the other cases presented here. A second aspect of interest is to compare the computational times associated with the different resolution techniques, that is, Nelson's method, residual iteration and subspace iteration techniques, as well as the PCPG method. The comparison is done with respect to the number of parameters studied and the number of target modes. Iterative techniques were performed with a tolerance of 10^{-6}, using convergence criterion (21).

Figures 7 and 8 represent the computational times of the different methods and the speedups $t_{\text{Nelson}}/t_{\text{ite}}$ with respect to the number of parameters studied (NP from 1 to 10), respectively, for the engine cover and Ariane 5 models.

Figures 9 and 10 represent, for the two same models, the computational times and the speedups with respect to the number of target modes (NM from 2 to 42).

Figures 7 and 8 indicate that the efficiency of the iterative methods with respect to Nelson's method decreases with increasing NP, even if the speedups are greater than 1 for the cases studied (until $NP = 10$). Indeed, in Nelson's method, the factorization of $Z(\omega_j)$ needed to solve Eq. (4) can be used independently for every parameter studied. On the contrary, the factorization of $Z_R(\omega_j)$ used in Eq. (16) has to be performed for each mode but also for each parameter of interest, because the reduction basis depends on this parameter. Similarly, the resolution by the PCPG method must be performed for each mode and each parameter of interest, even if the factorization of $\hat{K}$ can be kept. Nelson's method thus becomes more efficient, in terms of computational time, for increasing NP.

Figures 9 and 10 show that the speedups of the iterative methods strongly increase with the number of target modes NT for small NT and then slightly increase and finally decrease.

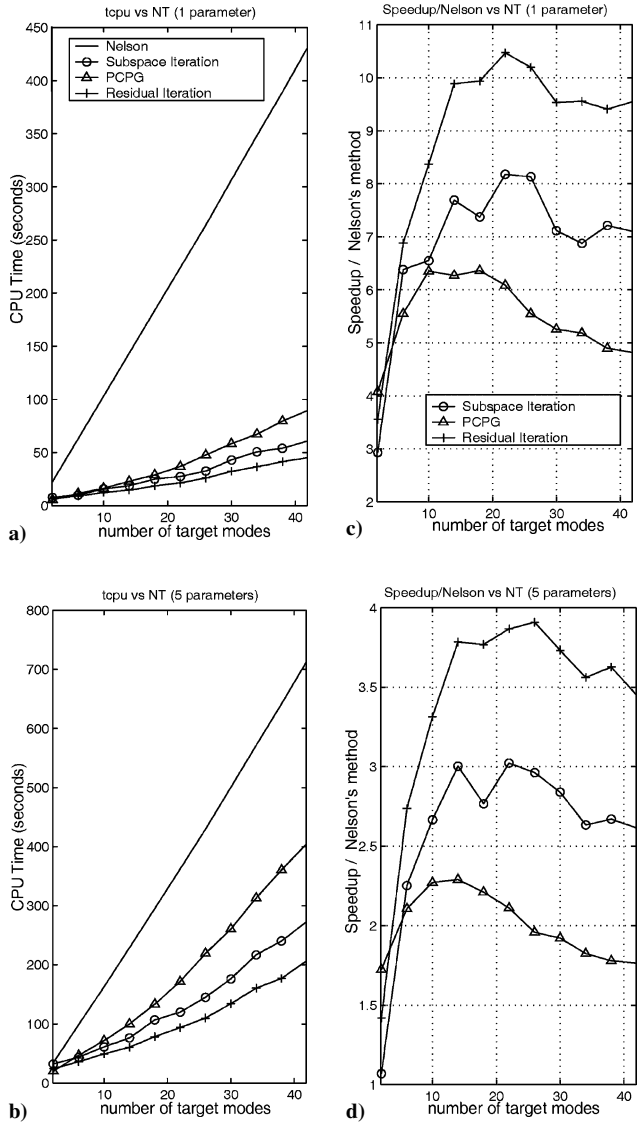

Fig. 9 Influence of target mode number (engine cover model): CPU times for a) one parameter and b) five parameters and speedup with respect to Nelson's method for c) one parameter and d) five parameters.

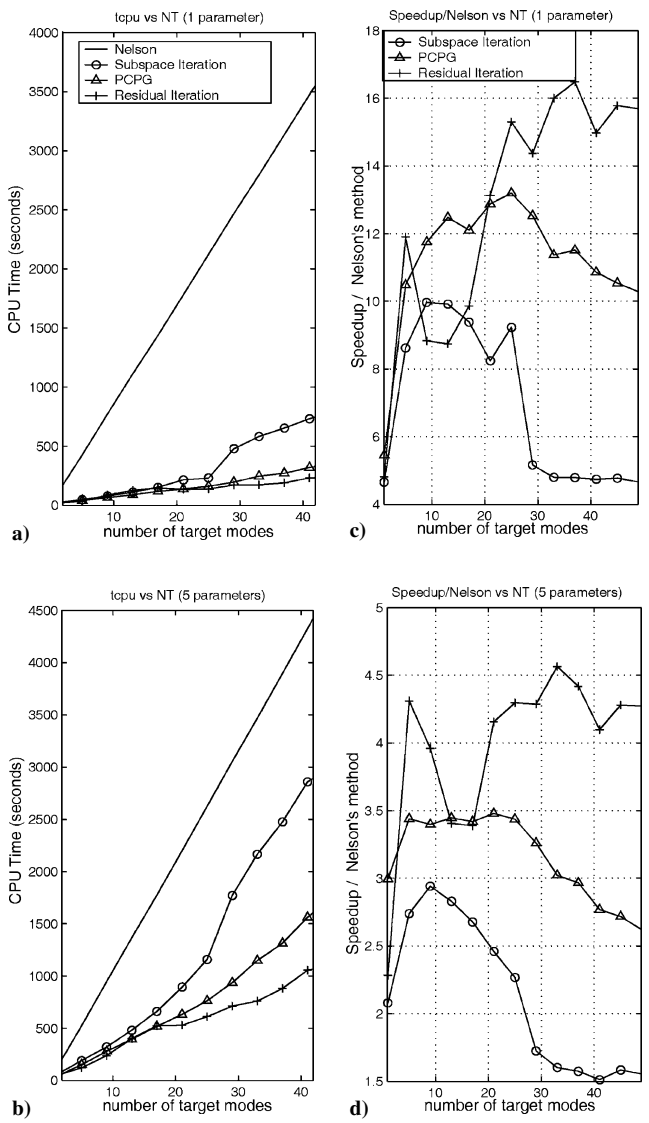

Fig. 10 Influence of target mode number (Ariane 5 model): CPU times for a) one parameter and b) five parameters and speedup with respect to Nelson's method for c) one parameter and d) five parameters.

In addition, speedups are greater for the Ariane 5 model than for the engine cover model. For the subspace and residual iteration methods, the main reason is that the Ariane 5 model is bigger, so that the ratio of the number of columns to the number of lines in the reduction basis NR/N is smaller. But the significant speedups for Ariane 5 (which reach 16.5) are also due to the fact that the oxygen tank contains a liquid. The fluid-structure coupling is modeled by added masses on the interface DOFs, leading to a full block of elements in the mass matrix. Thus the factorization of Z needed for Eq. (4) typically takes 70 s, whereas the factorization of K needed for Eq. (20) takes only around 26 s.

The speedup dependence to the number of DOFs N is strongly related to the factorization method. Indeed in the present study, using a sparse multifrontal factorization method developed by Boeing, the maximum speedup is around 16 for the computation of the sensitivities of 37 modes with respect to one parameter (Fig. 10c), whereas the speedup reached 72 for a less efficient skyline factorization method. The factorization of Z (Ariane 5) required 700 s with the skyline method and only 70 s with the sparse multifrontal method. If one used the only factorization methods available in MATLAB, the speedups would be much greater.

The residual iteration method was here nearly always the most efficient. The performance drop shown in Fig. 10 is linked to particular convergence problems that occasionally occur in all iterative methods.

IV. Conclusions

The paper provided a single framework allowing the comparison of most modeshape sensitivity approximation methods. It also introduced a residual iteration method to build an approximate basis for sensitivity predictions. The same method has also proved useful in other applications.[8–11]

Results on small industrial models showed that error indicators are close to true error levels and gave trends on how to properly apply the method. Keeping modes up to 1.2 times the last target frequency significantly improves convergence. Iterative methods are most interesting when a fairly large number of modes is considered and not too many parameters with speedups above 10 are common.

Application to larger models gave similar trends but was occasionally confronted to convergence problems. Eventually, the most interesting use of the residual iteration method is to allow very fast development of approximate solutions of good quality and low computational cost. One can thus trade accuracy for speedups of orders of magnitude in a controlled manner, and that tradeoff is often essential in design, optimization, or stochastic applications.

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